Math 155, *Lecture Notes- Bonds* **Name**____________

Section 8.4 Trigonometric Substitutions

When working with integrands that contain the following three expressions,

$$\sqrt{a^2-u^2},\ \sqrt{a^2+u^2},\ \text{and}\ \sqrt{u^2-a^2},$$

(difference of squares or sum of squares),

you should consider applying a trigonometric substitution technique. That is, re-write the integral using trigonometric functions based on a particular right triangle defined by the sides of u and a.

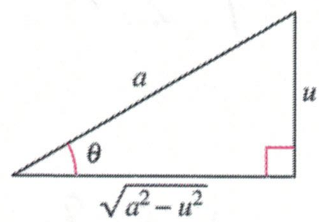
A critical component of this technique is to apply differentiation to our u-substitution function and solve for dx. You might notice that our examples, x is actually a function of θ, so we'll be solving for dx in terms of $d\theta$.

Trigonometric Substitution ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

$$u = a \sin \theta.$$

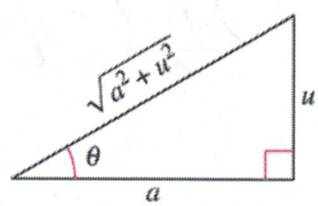
Then $\sqrt{a^2 - u^2} = a \cos \theta$, where $-\pi/2 \le \theta \le \pi/2$.



2. For integrals involving $\sqrt{a^2 + u^2}$, let

$$u = a \tan \theta.$$

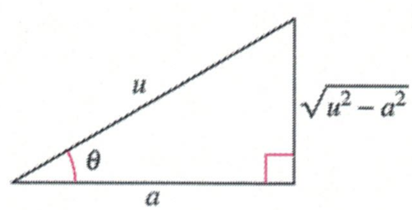
Then $\sqrt{a^2 + u^2} = a \sec \theta$, where $-\pi/2 < \theta < \pi/2$.



3. For integrals involving $\sqrt{u^2 - a^2}$, let

$$u = a \sec \theta.$$

Then $\sqrt{u^2 - a^2} = \pm a \tan \theta$, where $0 \le \theta < \pi/2$ or $\pi/2 < \theta \le \pi$. Use the positive value if $u > a$ and the negative value if $u < -a$.



NOTE: We will need to carefully create our right triangle in order to see all of the relevant trigonometric functions. Don't forget to label the side of your triangle using *opp*, *adj*, and *hyp*.

KEY: ★★ $\sqrt{9-x^2} = \sqrt{a^2-u^2}$

$a=3, \quad u=x$

Ex.1 Integrate: $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,dx$

$$= \int [\tan(\theta)]\cdot[3\cos(\theta)\,d\theta]$$

$$= 3\cdot\int \frac{\sin(\theta)}{\cos(\theta)}\cdot\frac{\cos(\theta)}{1}\,d\theta$$

$$= 3\int \sin(\theta)\,d\theta$$

$$= 3\cdot[-\cos(\theta)] + C$$

$$= -3\cdot\left[\frac{\sqrt{9-x^2}}{3}\right] + C$$

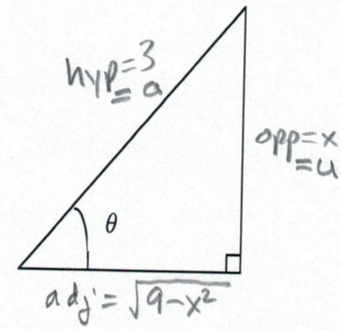
$$\boxed{= -\sqrt{9-x^2} + C}$$

check: $\dfrac{d}{dx}\left[-\sqrt{9-x^2} + C\right]$

$$= -\frac{d}{dx}\left[(9-x^2)^{1/2}\right] + 0$$

$$= -\frac{1}{2}\cdot(9-x^2)^{-1/2}\cdot(-2x)$$

$$= \frac{x}{\sqrt{9-x^2}} \quad ✓$$



Let $\dfrac{x}{3} = \dfrac{opp}{hyp} = \dfrac{u}{a}$

$\dfrac{x}{3} = \sin(\theta)$

$x = 3\sin(\theta)$

$\dfrac{dx}{d\theta} = 3\cos(\theta)$

$dx = \dfrac{dx}{d\theta}\cdot d\theta$

$dx = 3\cos(\theta)\,d\theta$

$\dfrac{x}{\sqrt{9-x^2}} = \dfrac{opp}{adj}$

$\dfrac{x}{\sqrt{9-x^2}} = \tan(\theta)$

$\cos(\theta) = \dfrac{adj}{hyp}$

$\cos(\theta) = \dfrac{\sqrt{9-x^2}}{3}$

Key: ★★ $\sqrt{4x^2+9} = \sqrt{a^2+u^2}$

$u^2 = 4x^2$ | $a^2 = 9$

$u = 2x$ | $a = 3$

Ex.2 Find the antiderivative: $\int \frac{\sqrt{4x^2+9}}{x^4}dx$

$$= \int \frac{[3\sec(\theta)]}{[\frac{3}{2}\tan(\theta)]^4} \cdot [\tfrac{3}{2}\sec^2(\theta)\,d\theta]$$

$$= \frac{9}{2} \cdot \int \frac{\sec^3(\theta)}{\frac{81}{16}\tan^4(\theta)} d\theta$$

$$= \frac{1}{\frac{9}{8}} \cdot \int \frac{\left[\frac{1}{\cos(\theta)}\right]^3}{\left[\frac{\sin(\theta)}{\cos(\theta)}\right]^4} d\theta$$

$$= \frac{8}{9} \int \frac{1}{\cos^3(\theta)} \cdot \frac{\cos^4(\theta)}{\sin^4(\theta)} d\theta$$

$$= \frac{8}{9} \int \frac{\cos(\theta)}{\sin^4(\theta)} d\theta$$

$$= \frac{8}{9} \int \frac{\cos(\theta)}{z^4} \cdot \left(\frac{dz}{\cos(\theta)}\right)$$

$$= \frac{8}{9} \int z^{-4} dz$$

$$= \frac{8}{9} \cdot \left[\frac{z^{-3}}{-3}\right] + C$$

$$= -\frac{8}{27 z^3} + C$$

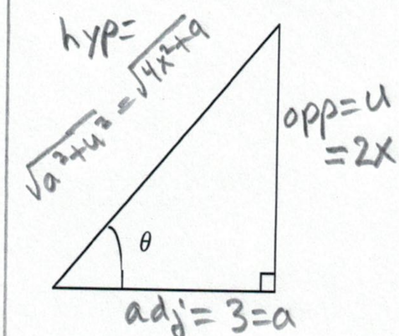
$$= -\frac{8}{27[\sin(\theta)]^3} + C$$

$$= \frac{-8}{27} \csc^3(\theta) + C$$

$$= \frac{-8}{27} \cdot \left[\frac{\sqrt{4x^2+9}}{2x}\right]^3 + C = \frac{-8}{27} \cdot \frac{(4x^2+9)^{3/2}}{8x^3}$$

$$\boxed{= -\frac{(4x^2+9)^{3/2}}{27x^3} + C}$$

check: ??



Let $\frac{u}{a} = \frac{opp}{adj}$

$\frac{2x}{3} = \tan(\theta)$

$2x = 3\tan(\theta)$

$x = \frac{3}{2}\tan(\theta)$

$\frac{d}{d\theta}(x) = \frac{d}{d\theta}[\frac{3}{2}\tan(\theta)]$

$\frac{dx}{d\theta} = \frac{3}{2}\sec^2(\theta)$

$dx = \frac{dx}{d\theta} \cdot d\theta$

$dx = \frac{3}{2}\sec^2(\theta)\,d\theta$

$\frac{\sqrt{4x^2+9}}{3} = \frac{hyp}{adj} = \sec(\theta)$

$\sqrt{4x^2+9} = 3\sec(\theta)$

Let $z = \sin(\theta)$

$\frac{dz}{d\theta} = \cos(\theta)$

$\frac{dz}{\cos(\theta)} = d\theta$

$\csc(\theta) = \frac{hyp}{opp}$

$\csc(\theta) = \frac{\sqrt{4x^2+9}}{2x}$

KEY: ★★ $\sqrt{x^2-9}=\sqrt{u^2-a^2}$

$u=x\ ,\ a=3$

4/6

Ex.3 Evaluate: $\displaystyle\int_3^6 \frac{\sqrt{x^2-9}}{x^2}dx = \int_3^6 \left[\frac{\sqrt{x^2-9}}{x}\right]\cdot\left[\frac{1}{x}\right]dx$

$$= \int_{\theta=0}^{\theta=\pi/3} [\sin(\theta)]\left[\frac{1}{3\sec(\theta)}\right]\cdot[3\sec(\theta)\tan(\theta)\,d\theta]$$

$$= \int_0^{\pi/3} [\sin(\theta)]\cdot\left[\frac{\sin(\theta)}{\cos(\theta)}\right]d\theta$$

$$= \int_0^{\pi/3} \frac{\sin^2(\theta)}{\cos(\theta)}d\theta$$

$$= \int_0^{\pi/3} \frac{1-\cos^2(\theta)}{\cos(\theta)}d\theta$$

$$= \int_0^{\pi/3} \frac{1}{\cos(\theta)}d\theta - \int_0^{\pi/3} \frac{\cos^2(\theta)}{\cos(\theta)}d\theta$$

$$= \int_0^{\pi/3} \sec(\theta)d\theta - \int_0^{\pi/3} \cos(\theta)d\theta$$

$$= \Big[\ln|\sec(\theta)+\tan(\theta)|\Big]_0^{\pi/3} - \Big[\sin(\theta)\Big]_0^{\pi/3}$$

$$= \ln|\sec(\tfrac{\pi}{3})+\tan(\tfrac{\pi}{3})| - \ln|\sec(0)+\tan(0)| \ ★$$
$$- \left[\sin(\tfrac{\pi}{3}) - \sin(0)\right]$$

$$= \Big[\ln|2+\sqrt{3}| - \ln|1+0|\Big] - \left[\frac{\sqrt{3}}{2} - 0\right]$$

$$= \ln(2+\sqrt{3}) - \ln(1) - \frac{\sqrt{3}}{2}$$

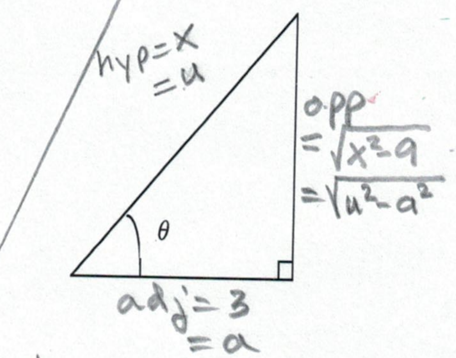
$$\boxed{= \ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}}$$

★ $\sec(\frac{\pi}{3}) = \dfrac{1}{\cos(\frac{\pi}{3})} = \dfrac{1}{1/2} = 2$

$\tan(\frac{\pi}{3}) = \dfrac{\sin(\frac{\pi}{3})}{\cos(\frac{\pi}{3})} = \dfrac{\sqrt{3}/2}{1/2} = \sqrt{3}$

$\sec(0)=1$

$\tan(0)=0$



Let

$\dfrac{u}{a} = \dfrac{x}{3} = \dfrac{hyp}{adj}$

$\dfrac{x}{3} = \sec(\theta)$

$x = 3\sec(\theta)$

$\dfrac{dx}{d\theta} = 3\sec(\theta)\tan(\theta)$

$dx = \dfrac{dx}{d\theta}\cdot d\theta$

$dx = 3\sec(\theta)\tan(\theta)\,d\theta$

$\dfrac{\sqrt{x^2-9}}{x} = \dfrac{opp}{hyp}$

$\dfrac{\sqrt{x^2-9}}{x} = \sin(\theta)$

change variables:

$x=3\sec(\theta)\ ,\ x=3$

$3=3\sec(\theta)$

$1=\sec(\theta)=\dfrac{1}{\cos(\theta)}$

$\cos(\theta)=1$

$\theta=0$

$x=3\sec(\theta)\ ,\ x=6$

$6=3\sec(\theta)$

$2=\sec(\theta)=\dfrac{1}{\cos(\theta)}$

$\cos(\theta)=\dfrac{1}{2}$

$\theta=\dfrac{\pi}{3}$

KEY: ★★ $\sqrt{x^2-4} = \sqrt{u^2-a^2}$

$x = u,\ a = 2$

Ex.4 Integrate: $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

$$= \int [\sin(\theta)] \cdot [2\sec(\theta)\tan(\theta)\,d\theta]$$

$$= 2\int \left(\frac{\sin(\theta)}{1}\right)\left(\frac{1}{\cos(\theta)}\right) \cdot [\tan(\theta)]\,d\theta$$

$$= 2\int [\tan(\theta)] \cdot [\tan(\theta)]\,d\theta$$

$$= 2\int \tan^2(\theta)\,d\theta$$

$$= 2\int [\sec^2(\theta) - 1]\,d\theta$$

$$= 2\cdot\int \sec^2(\theta)\,d\theta - 2\cdot\int 1\,d\theta$$

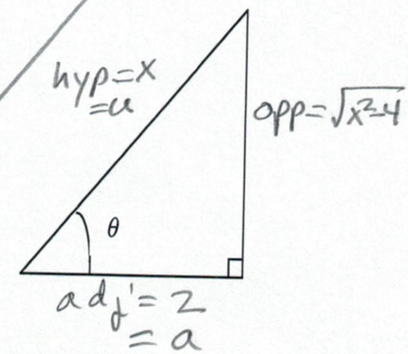
$$= 2\cdot[\tan(\theta)] - 2\cdot[\theta] + C$$

convert to "x"

$$= 2\cdot\left[\frac{\sqrt{x^2-4}}{2}\right] - 2\cdot\left[\operatorname{arcsec}\left(\frac{x}{2}\right)\right] + C$$

$$\boxed{= \sqrt{x^2-4} - 2\operatorname{arcsec}\left(\frac{x}{2}\right) + C}$$

check: ??



Let $\dfrac{u}{a} = \dfrac{x}{2} = \dfrac{hyp}{adj'}$

$\dfrac{x}{2} = \sec(\theta)$

$x = 2\sec(\theta)$

$\dfrac{d}{d\theta}(x) = \dfrac{d}{d\theta}[2\sec(\theta)]$

$\dfrac{dx}{d\theta} = 2\sec(\theta)\tan(\theta)$

$dx = 2\sec(\theta)\tan(\theta)\,d\theta$

$\dfrac{\sqrt{x^2-4}}{x} = \dfrac{opp}{hyp}$

$\dfrac{\sqrt{x^2-4}}{x} = \sin(\theta)$

use

$\tan^2(\theta) = \sec^2(\theta) - 1$

$\tan(\theta) = \dfrac{opp}{adj'}$

$\tan(\theta) = \dfrac{\sqrt{x^2-4}}{2}$

$\dfrac{x}{2} = \sec(\theta)$

$\operatorname{arcsec}\left(\dfrac{x}{2}\right) = \operatorname{arcsec}[\sec(\theta)]$

$\operatorname{arcsec}\left(\dfrac{x}{2}\right) = \theta$

KEY: ★★ $\sqrt{4-x^2}=\sqrt{a^2-u^2}$

$a=2,\ u=x$

Ex.5 Find the antiderivative: $\int \sqrt{16-4x^2}\,dx = \int \sqrt{4}\cdot\sqrt{4-x^2}\,dx$

$= 2\cdot\int\sqrt{4-x^2}\,dx$

$= 2\cdot\int[2\cos(\theta)]\cdot[2\cos(\theta)\,d\theta]$

$= 8\cdot\int\cos^2(\theta)\,d\theta$

$= 8\cdot\int\left[\frac{1+\cos(2\theta)}{2}\right]d\theta$

$= 4\cdot\int[1+\cos(2\theta)]\,d\theta$

$= 4\int 1\,d\theta + 4\cdot\int\cos(2\theta)\,d\theta$

$= 4\cdot[\theta] + 4\int[\cos(z)]\cdot\left(\frac{dz}{2}\right)$

$= 4\theta + 2\cdot\int\cos(z)\,dz$

$= 4\theta + 2\cdot[\sin(z)] + C$

$= 4\theta + 2\sin(2\theta) + C$

$= 4\theta + 2\cdot[2\sin(\theta)\cos(\theta)] + C$

$= 4\theta + 4\sin(\theta)\cos(\theta) + C$

$= 4\cdot\left[\arcsin\left(\frac{x}{2}\right)\right] + 4\cdot\left(\frac{x}{2}\right)\cdot\left(\frac{\sqrt{4-x^2}}{2}\right) + C$

$\boxed{= 4\arcsin\left(\frac{x}{2}\right) + x\sqrt{4-x^2} + C}$

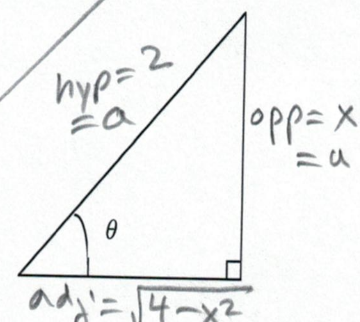
Check:

??

$\frac{x}{2} = \sin(\theta)$

$\arcsin\left(\frac{x}{2}\right) = \arcsin[\sin(\theta)]$

$\arcsin\left(\frac{x}{2}\right) = \theta$



Let $\frac{u}{a} = \frac{opp}{hyp} = \frac{x}{2}$

$\frac{x}{2} = \sin(\theta)$

$x = 2\sin(\theta)$

$\frac{dx}{d\theta} = 2\cos(\theta)$

$dx = 2\cos(\theta)\,d\theta$

$\frac{\sqrt{4-x^2}}{2} = \frac{adj}{hyp}$

$\frac{\sqrt{4-x^2}}{2} = \cos(\theta)$

$\sqrt{4-x^2} = 2\cos(\theta)$

$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

Let $z = 2\theta$

$\frac{dz}{d\theta} = 2$

$\frac{dz}{2} = d\theta$